Nonlinear current response of interacting fermions in metallic and insulating regimes

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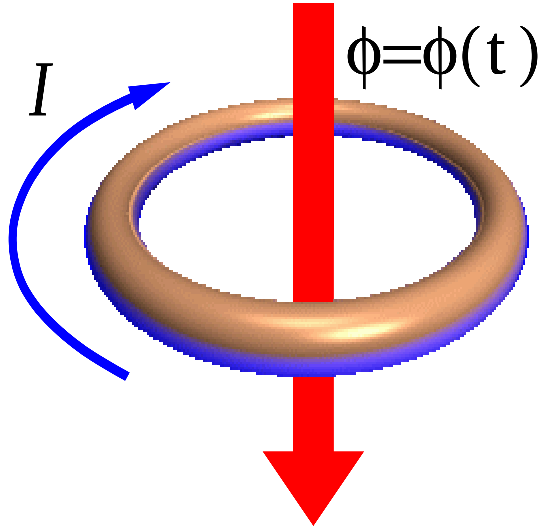
Motivation

Metallic and insulating regimes

- flux $\Phi(t) \longrightarrow$ electric field $F \propto \partial_t \Phi(t)$
- isolated 1D system: $\partial_t \rho(t) = -i[H(t), \rho(t)]$
- initially microcanonical ensemble with $\beta \ll 1$



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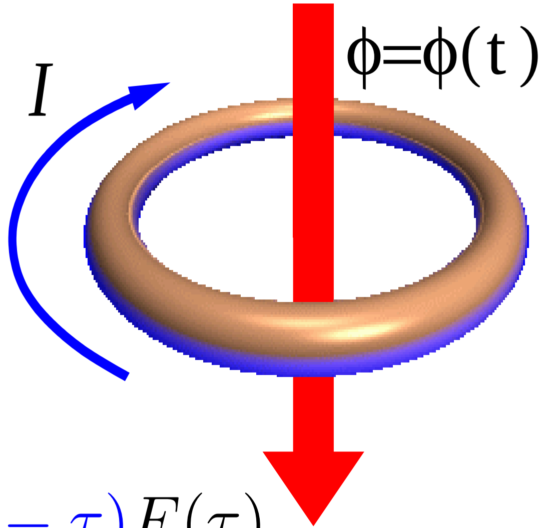

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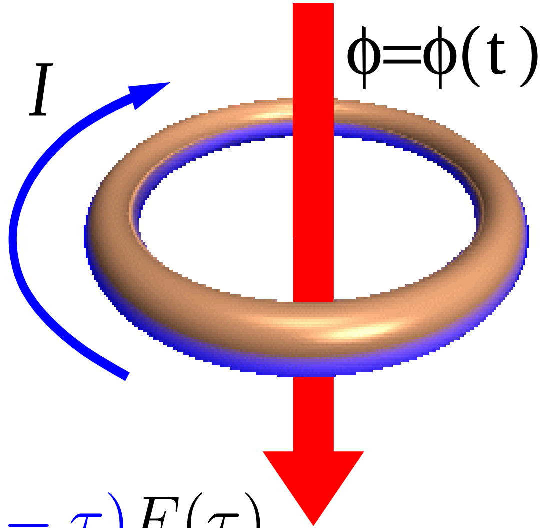

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Integrable vs. non–integrable systems for final F

- for $F = 0$: integrability $\longrightarrow$ conserved quantities $\longrightarrow$ relaxation
- in LR difference visible since response functions calculated at $F = 0$
- how the integrability–related properties of the LR change with F

t–V–W model

$$H = -t_h \sum_j \left\{ \mathrm{e}^{i\phi(t)}\, c^\dagger_{j+1} c_j + \mathrm{h.}c. \right\} + V \sum_j \hat{n}_j \hat{n}_{j+1} + W \sum_j \hat{n}_j \hat{n}_{j+2}$$

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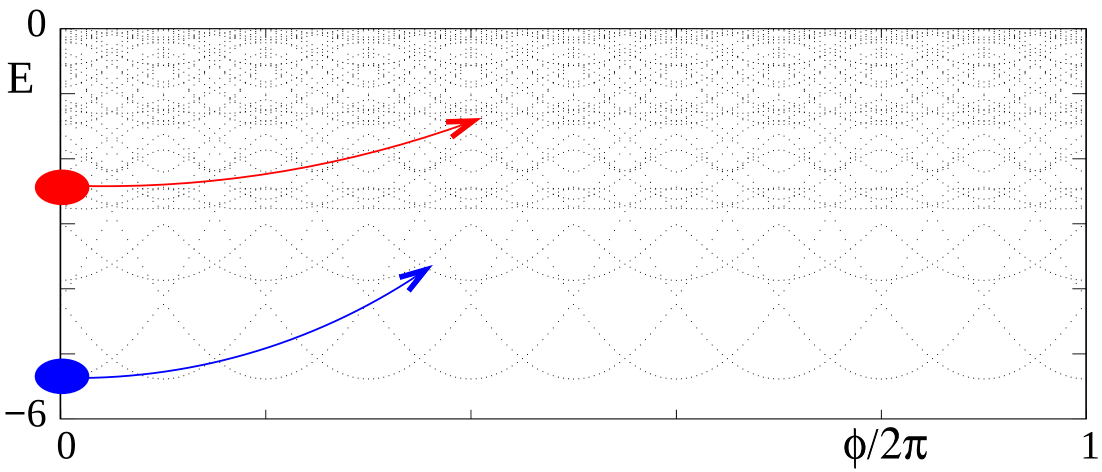

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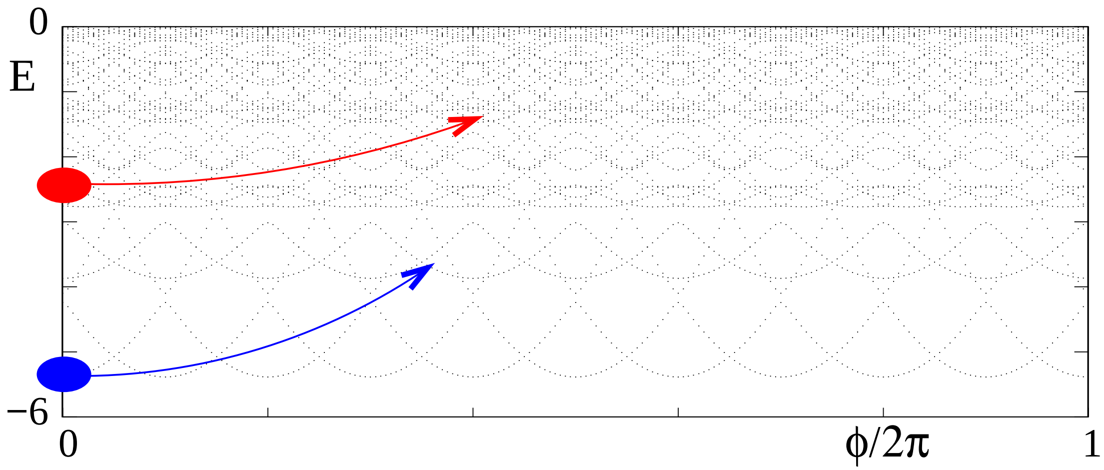


- initial $|\psi(0)\rangle$ with assumed energy $\bar{E}_0$ from MCLM, $L = 28$:

$$[H(0) - \bar{E}_0]^2 |\psi(0)\rangle = \varepsilon |\psi(0)\rangle$$

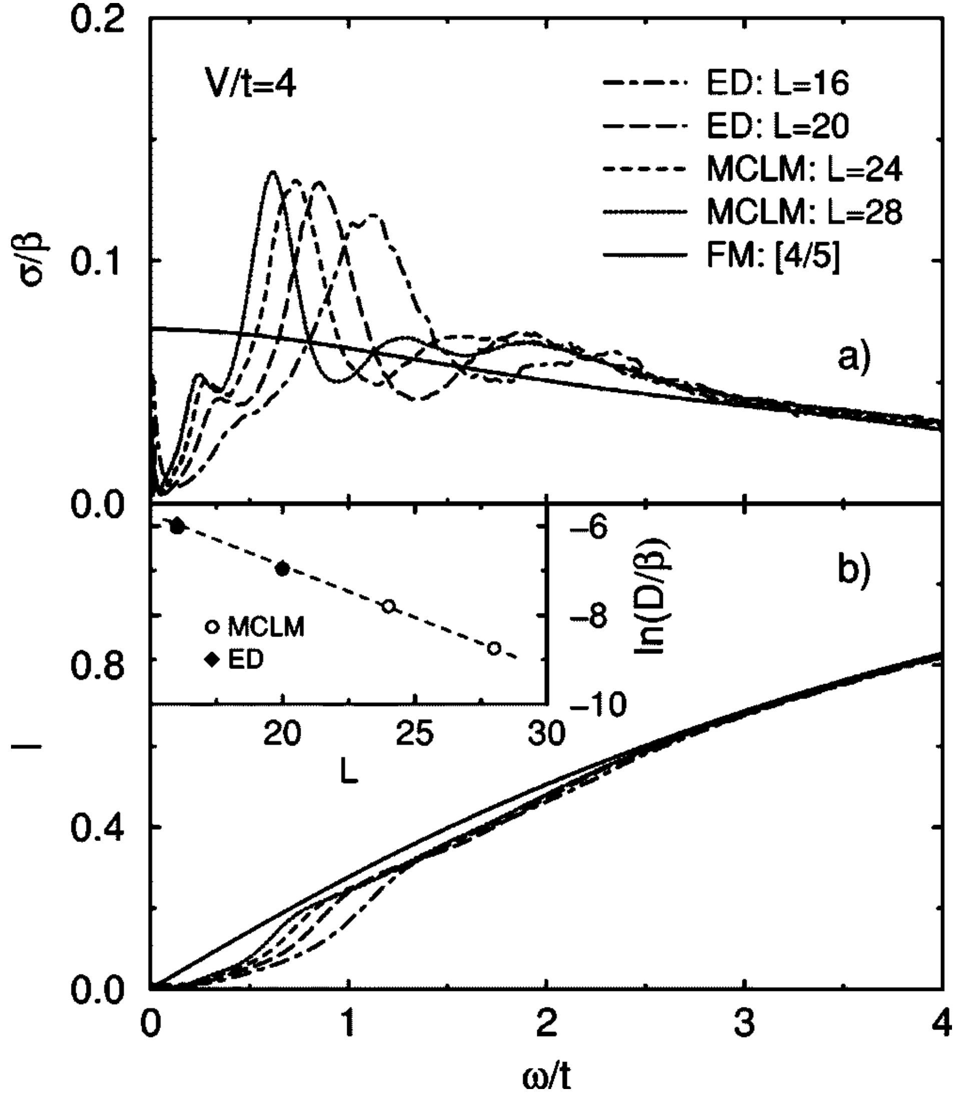
- $i\partial_t |\psi(t)\rangle = H[\phi(t)] |\psi(t)\rangle$, Park, Light, J. Chem. Phys. (1986)

Driving insulators, V>2

Challenges

- Involved finite–size scaling

P.Prelovšek *et al*, PRB 2004



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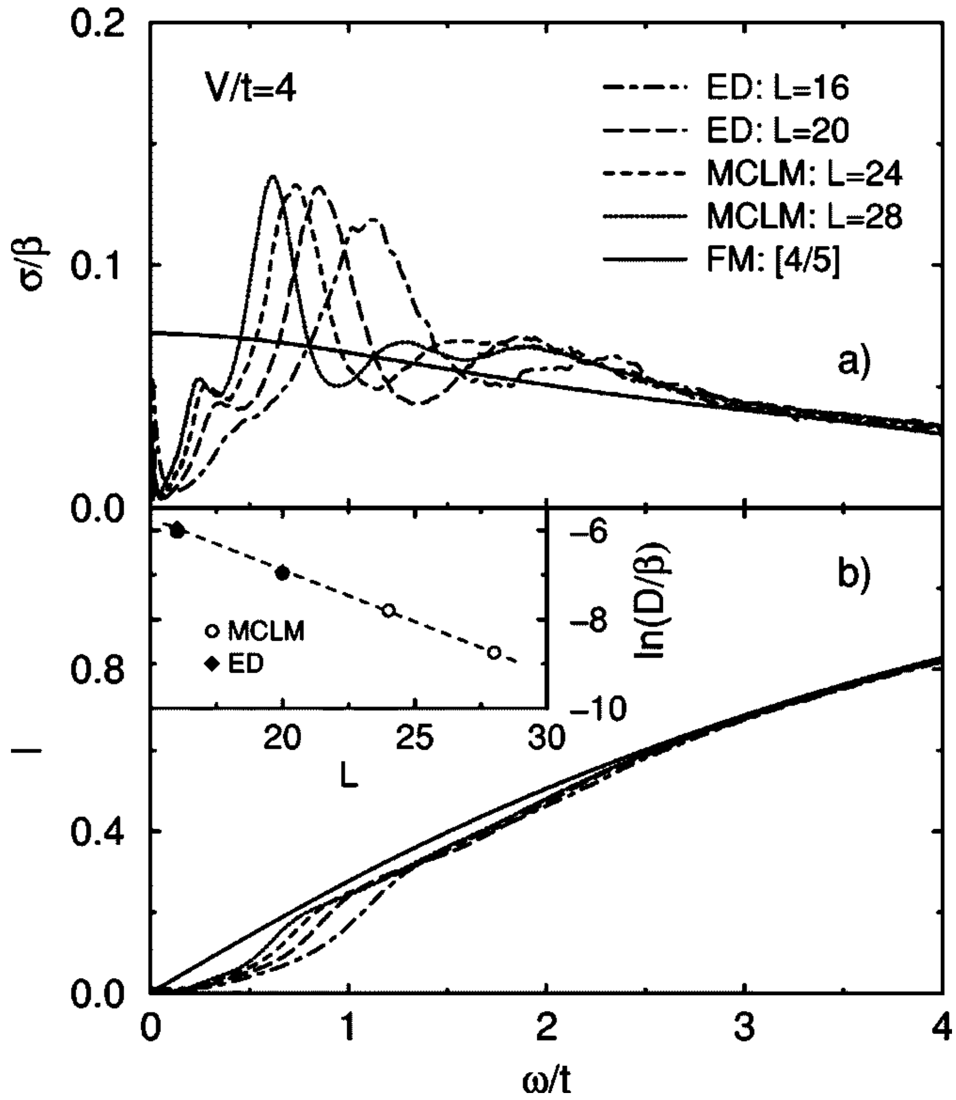

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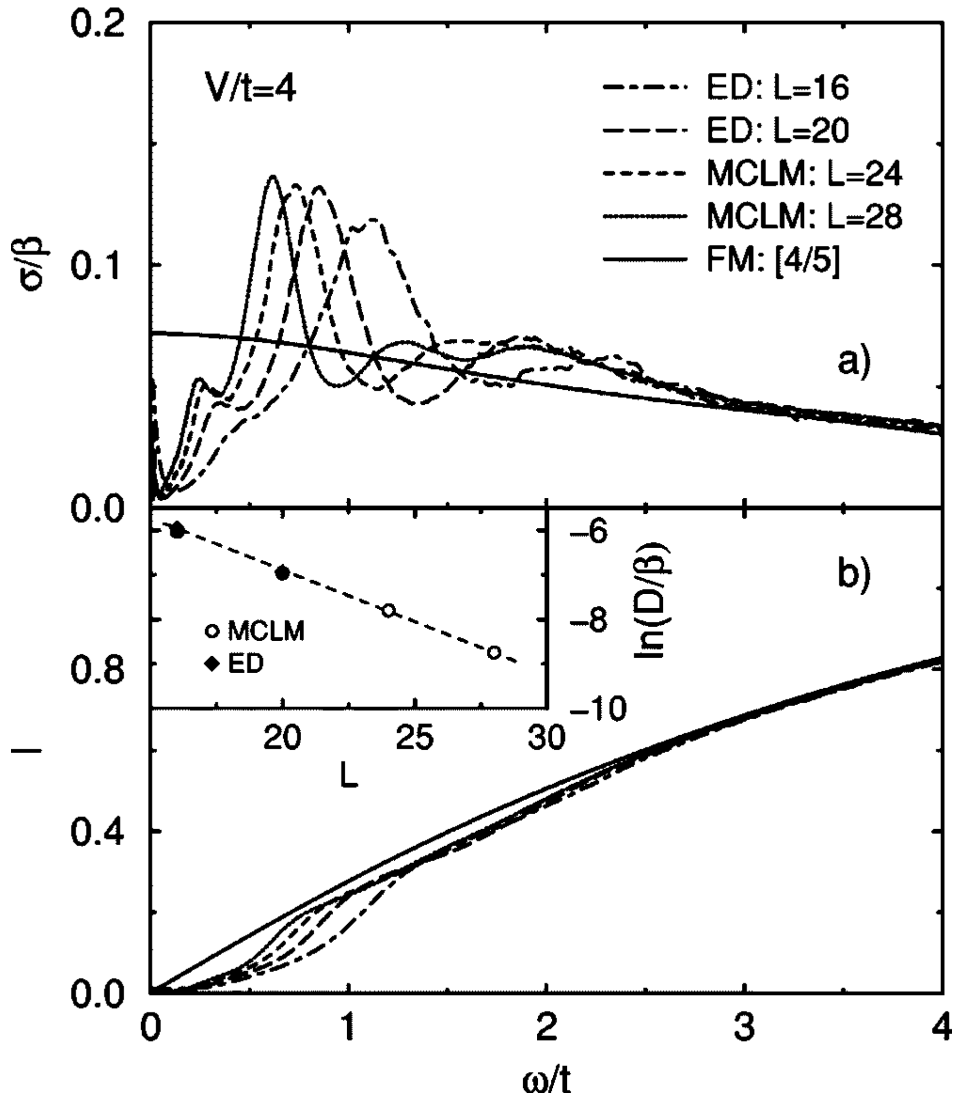
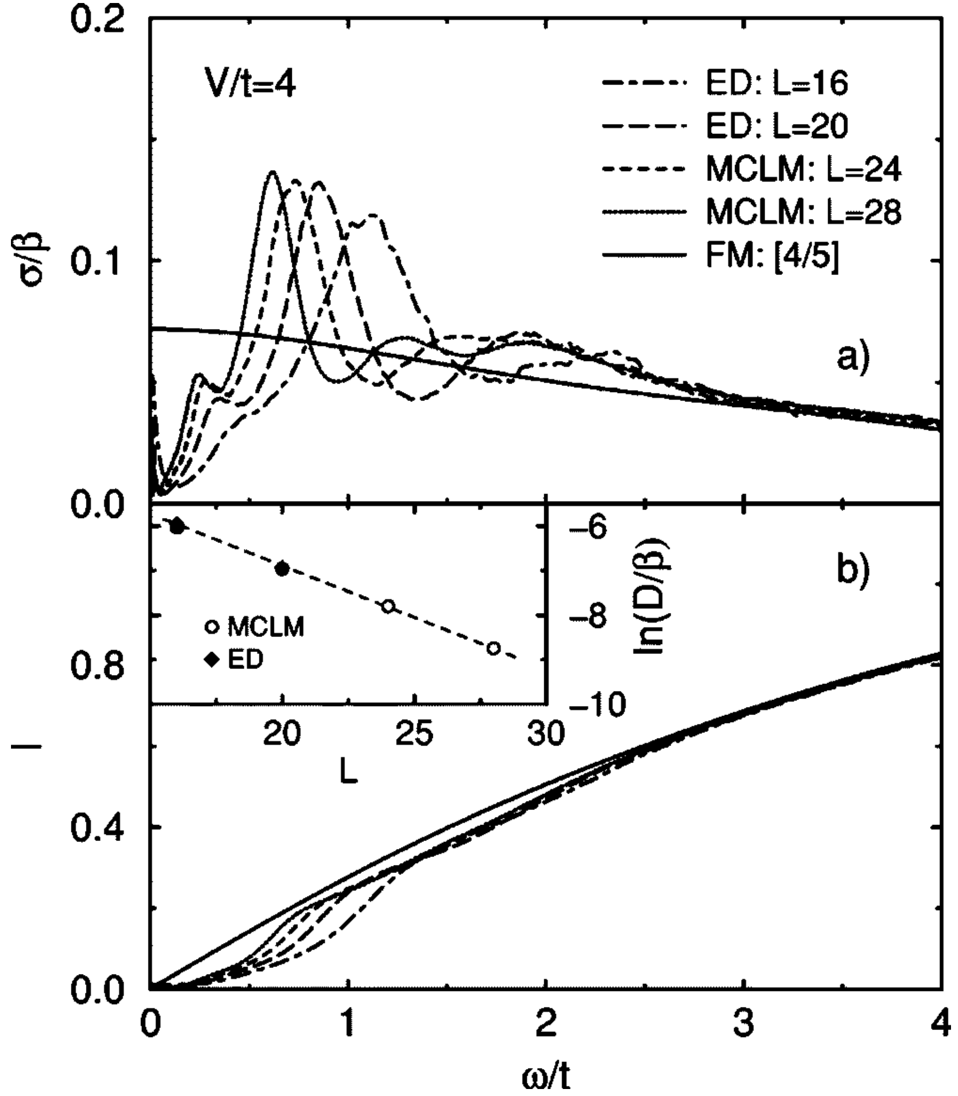


Driving insulators, V>2

Challenges

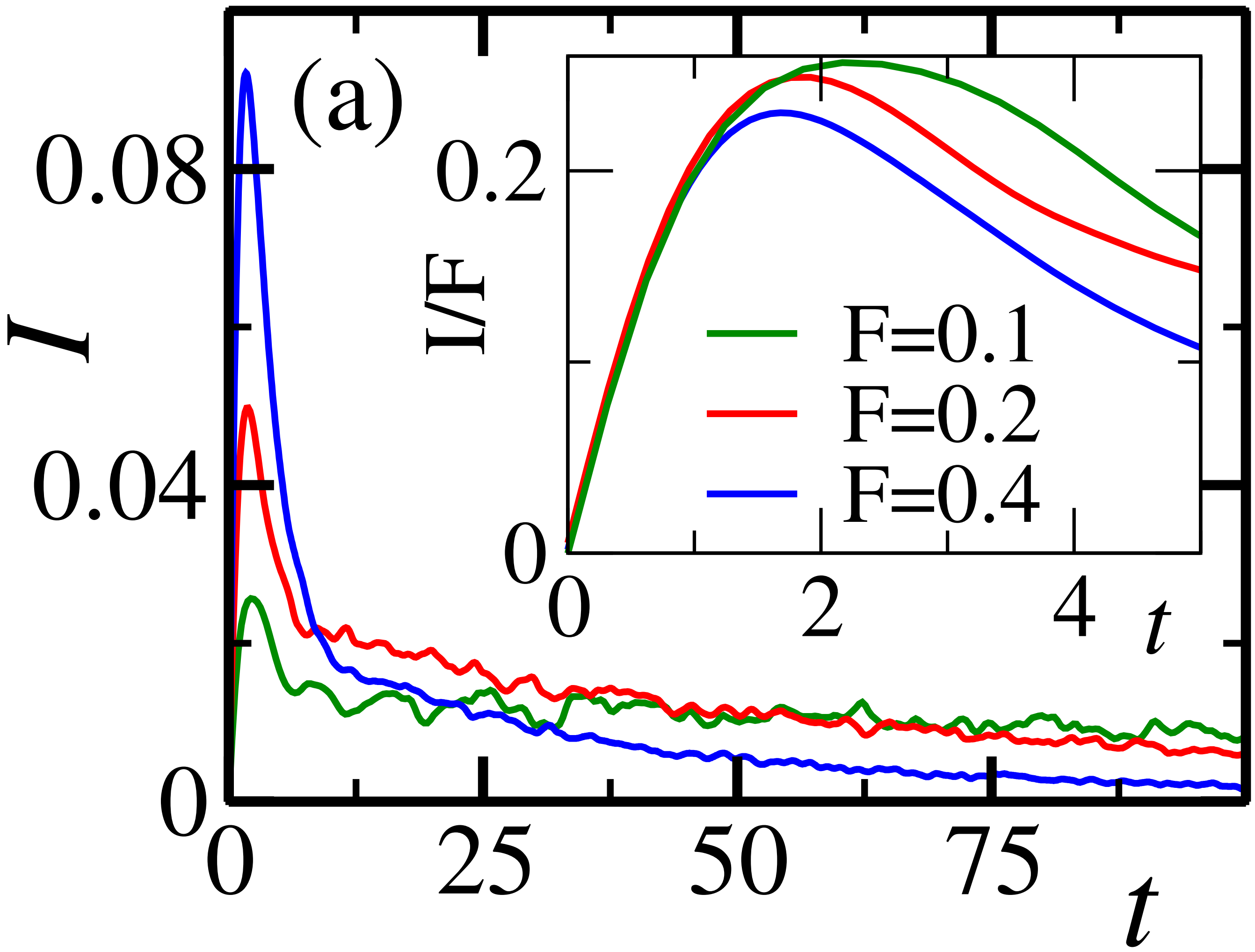
- Involved finite–size scaling
- Literally dc response expected for open quantum systems
- Coupling to leads may break integrability
- Driving may cause inhomogeneous distribution of carriers and destroy Mott insulating state

P.Prelovšek *et al*, PRB 2004



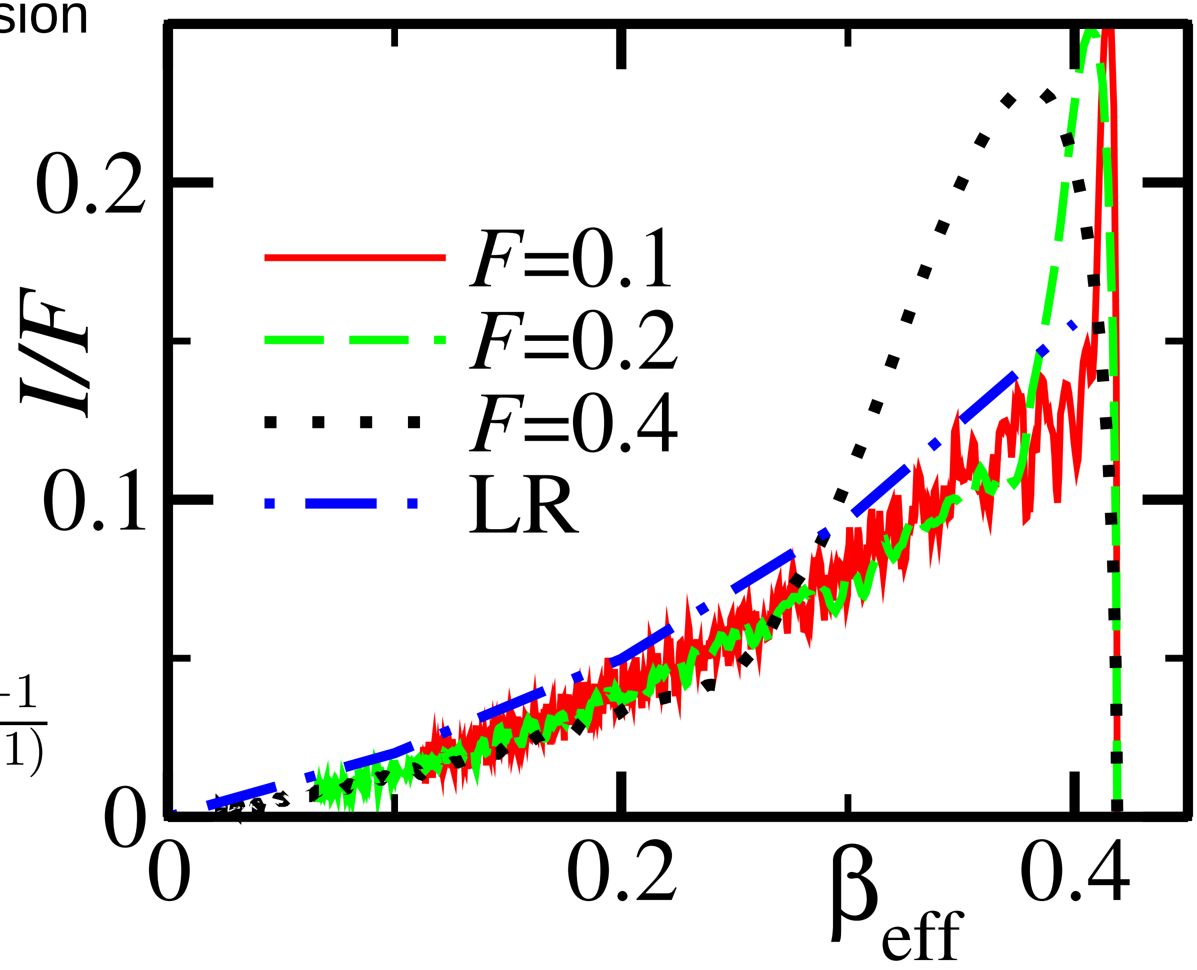
Limits of LR in NI insulator

Real–time current non–integrable (NI) case : $V = 3$, $W = 1$, $L = 26$



Subtracting heating - NI insulator

High–temperature expansion



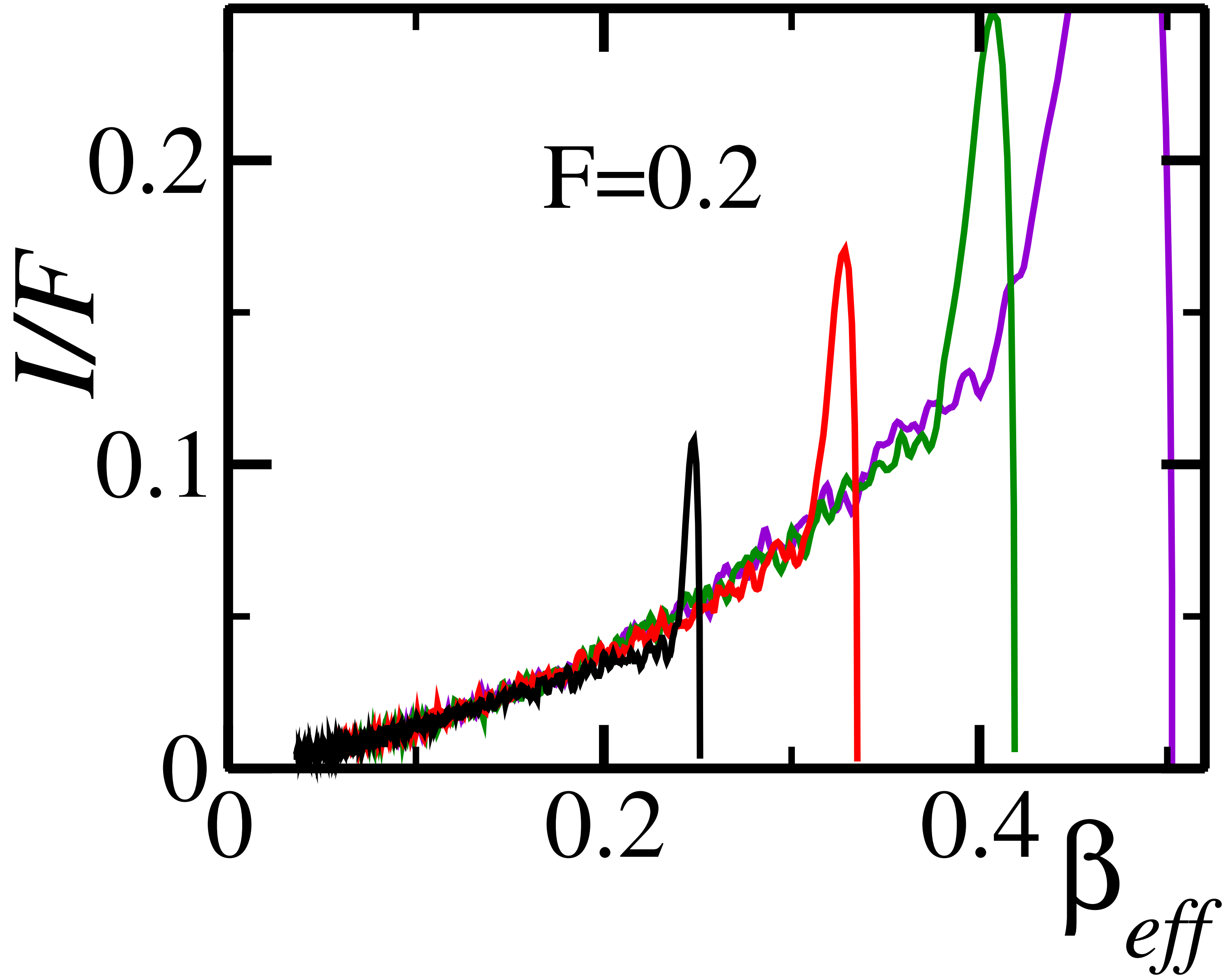
$$\varepsilon(t) = [E_\infty - E(t)]/L$$

$$E_\infty = L(V+W)\frac{L/2-1}{2(L-1)}$$

$$\frac{\varepsilon}{\beta} = \frac{1}{2} + \frac{V^2+W^2}{16}$$

$I(t) > 0 \rightarrow \beta_{eff}(t)$ is monotonic

Does $E(t)$ determine $I(t)$ in NI case?



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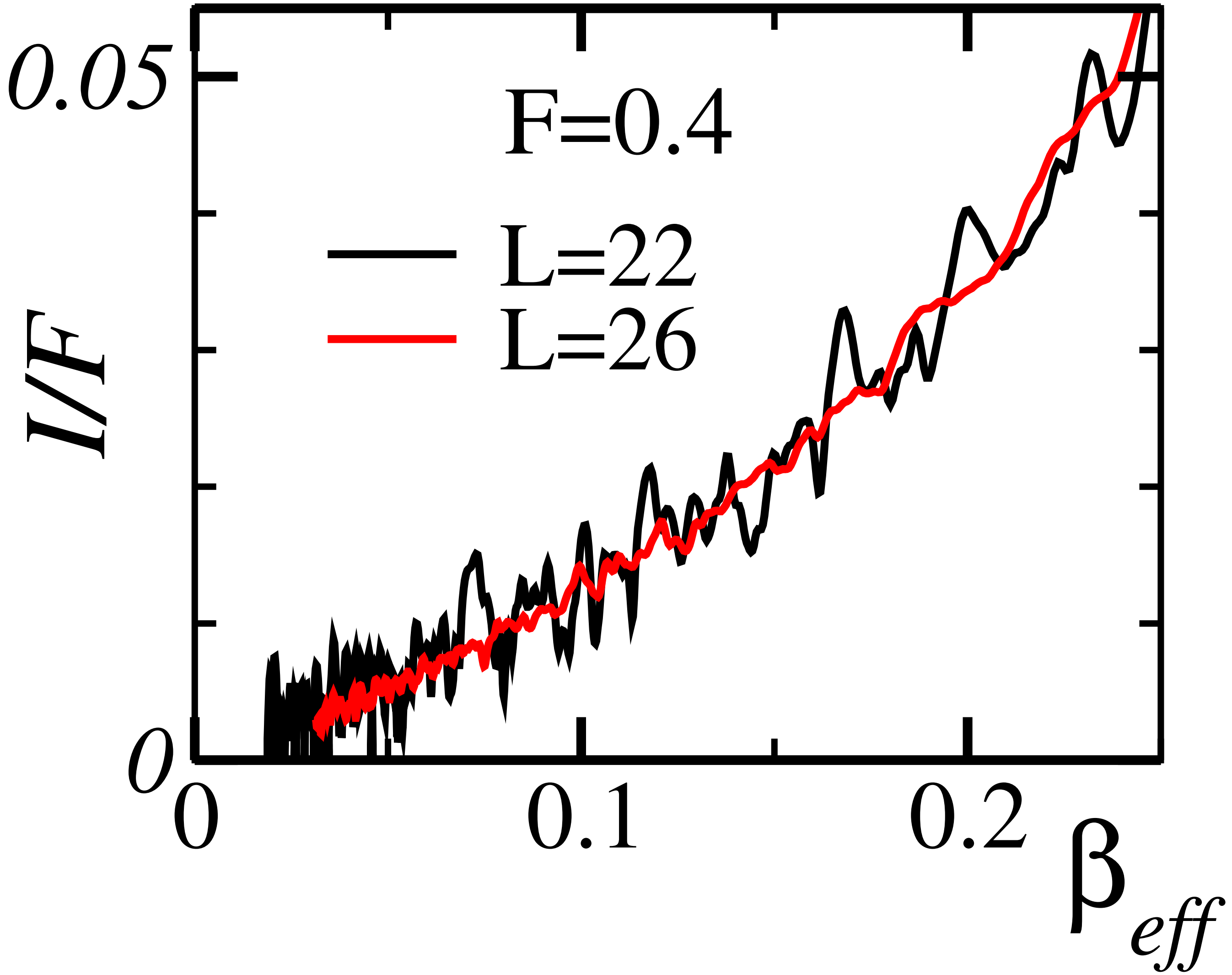
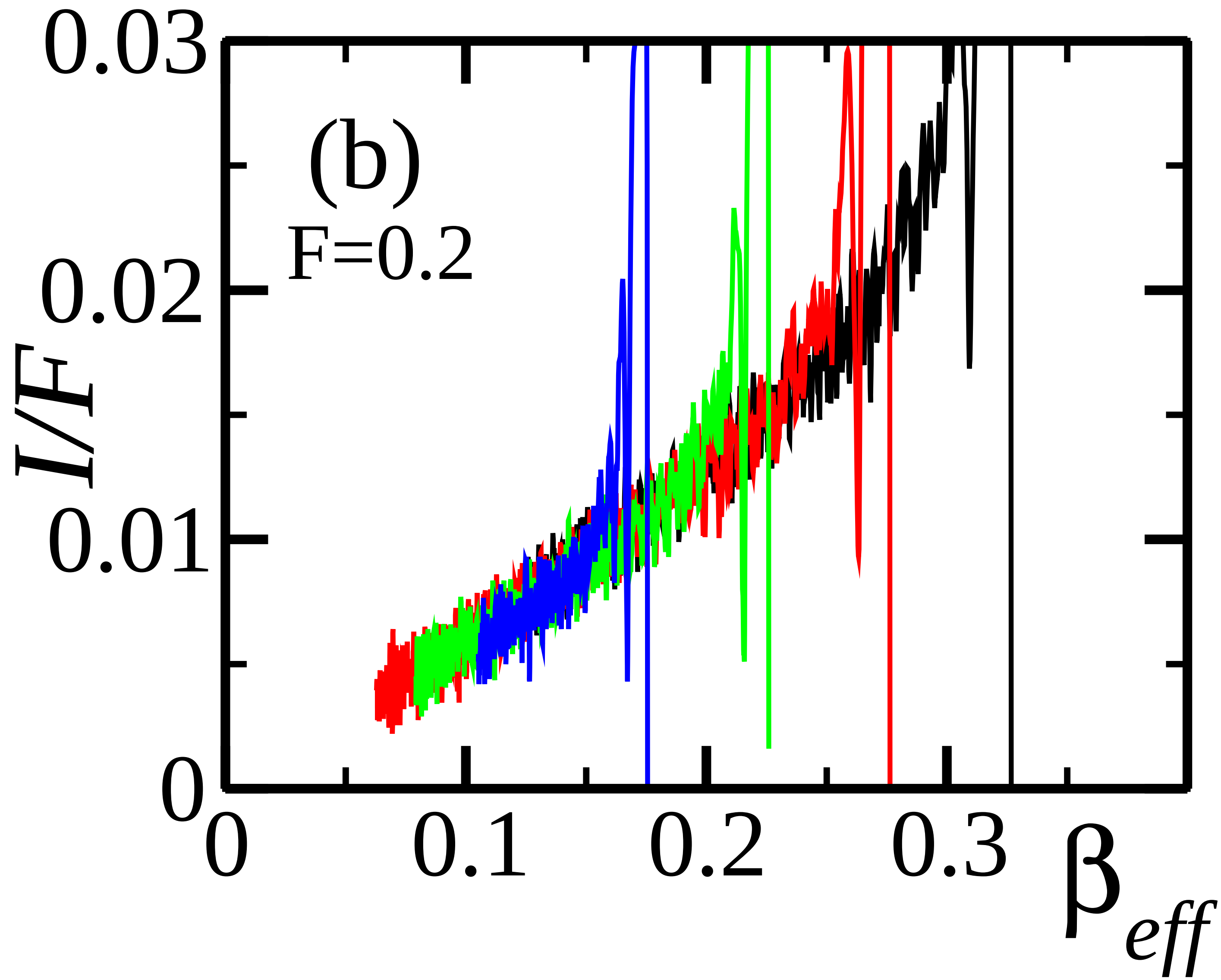


Integrable case and non–zero F

Integrable (I) case: $V = 3$, $W = 0$, $L = 28$



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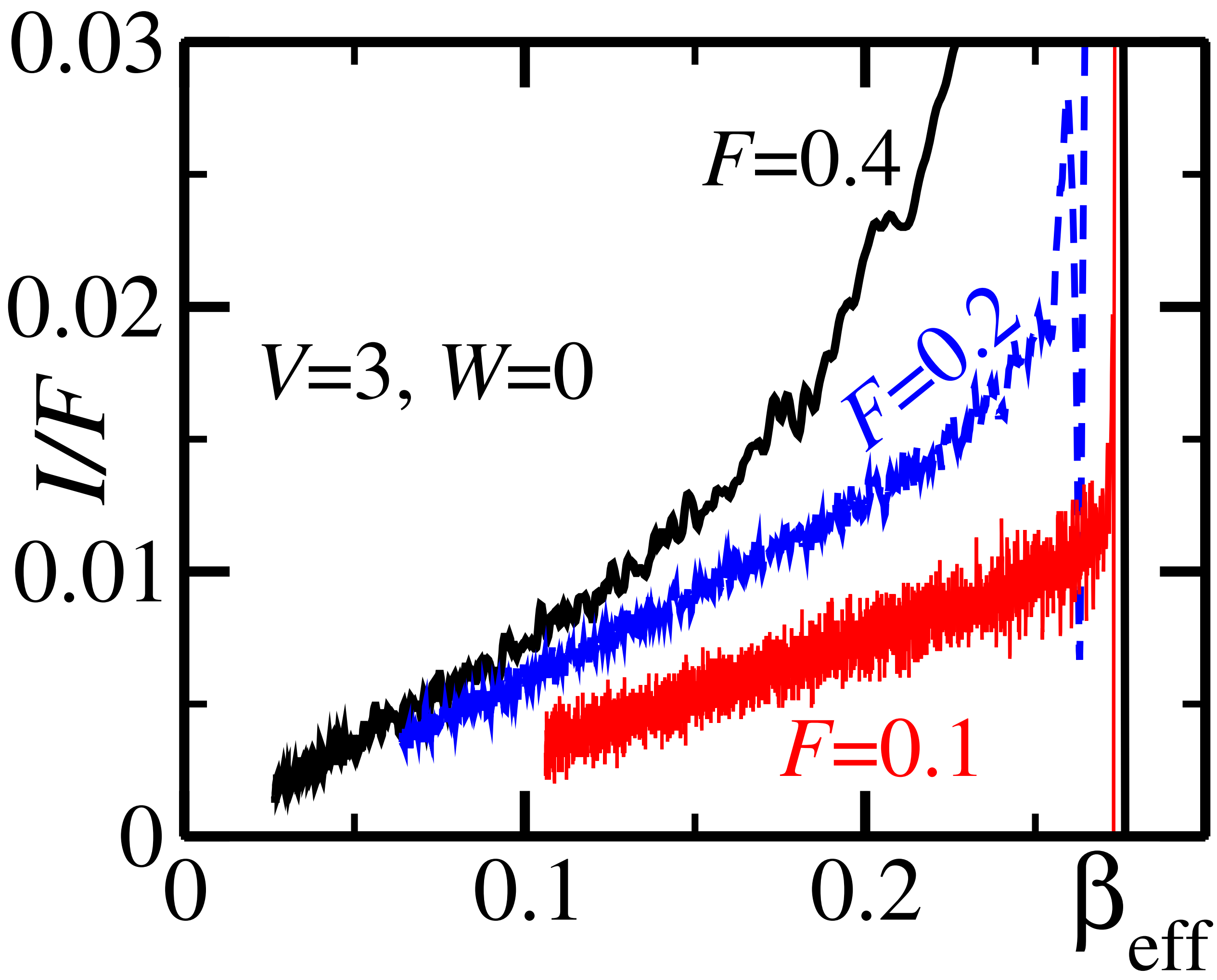
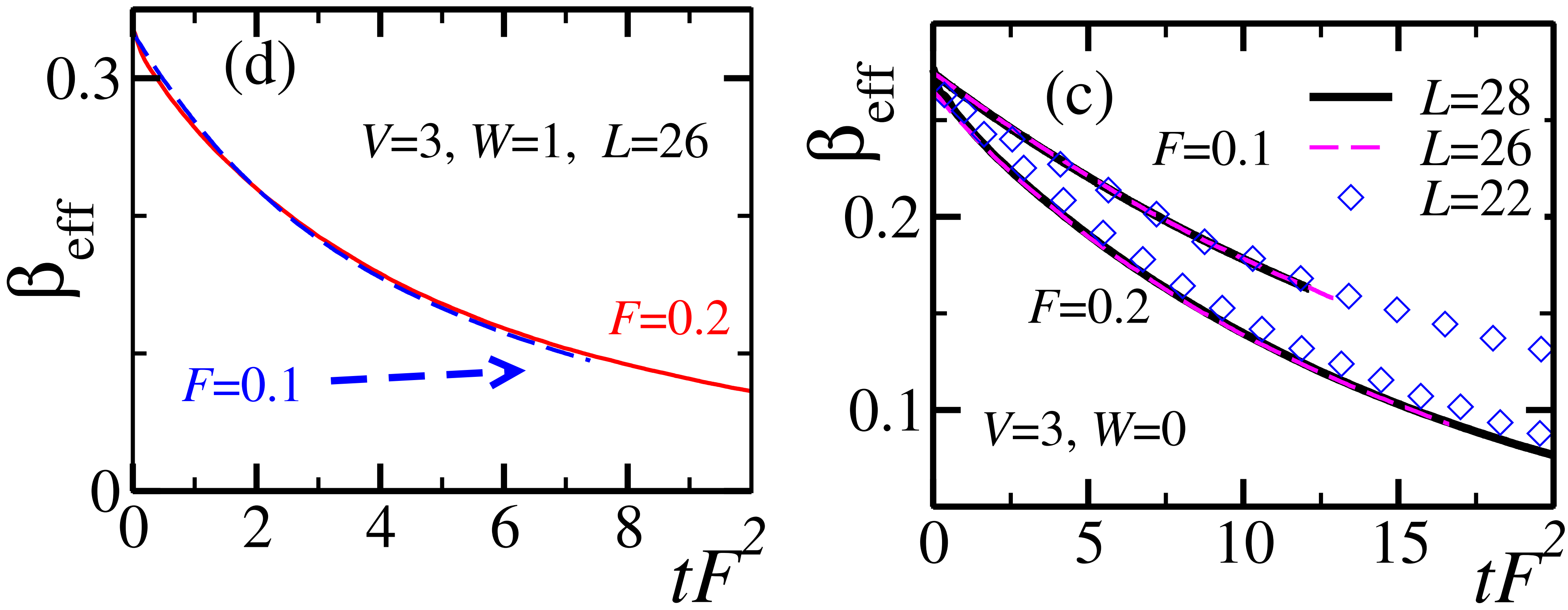


dc response from $E(t)$

$$\frac{d}{dt}E = L\ F\ I$$

$$\text{if}\quad I(E,F)/F = \gamma(E) \quad \text{then} \quad \frac{d}{dt}E \sim F^2$$



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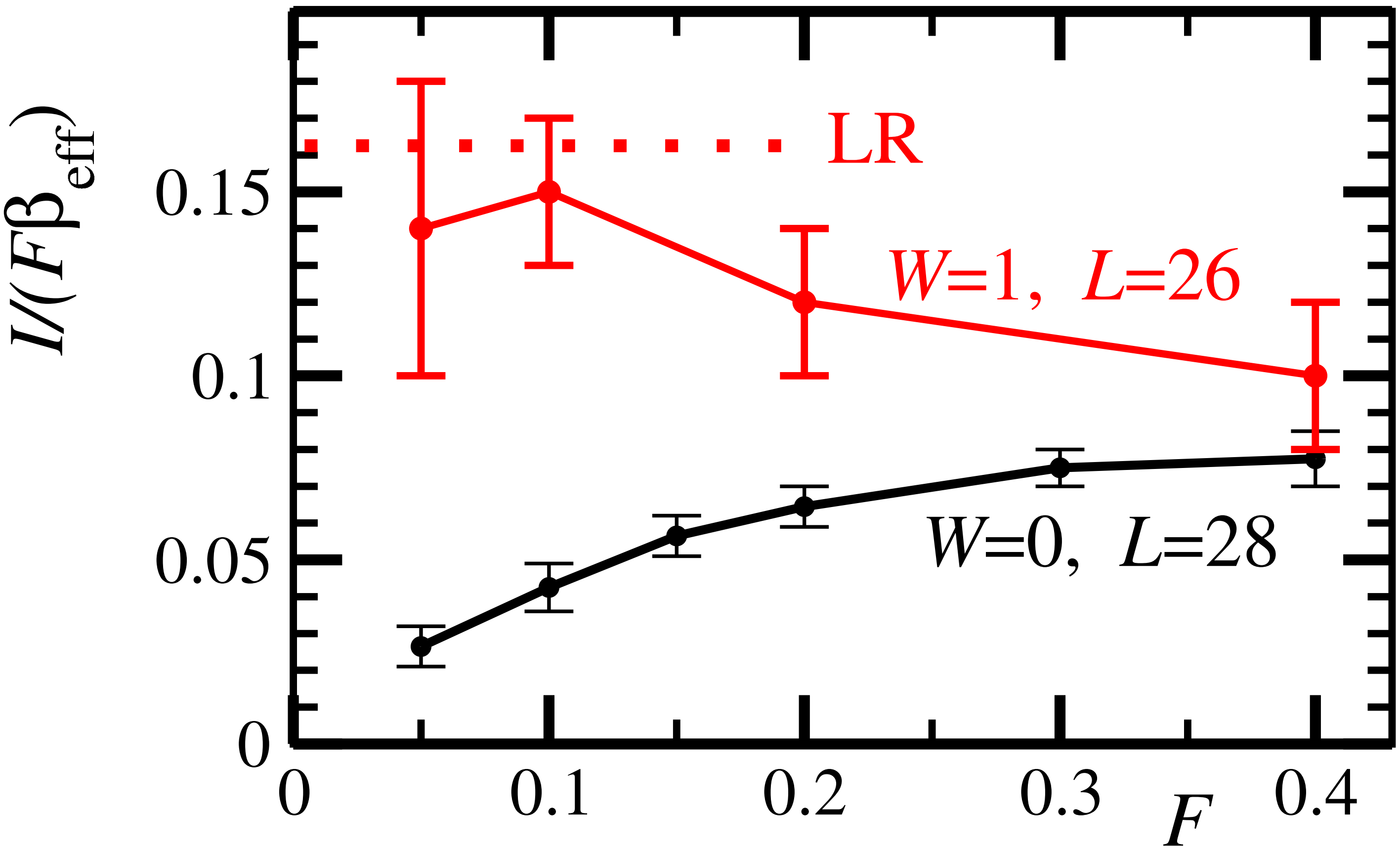
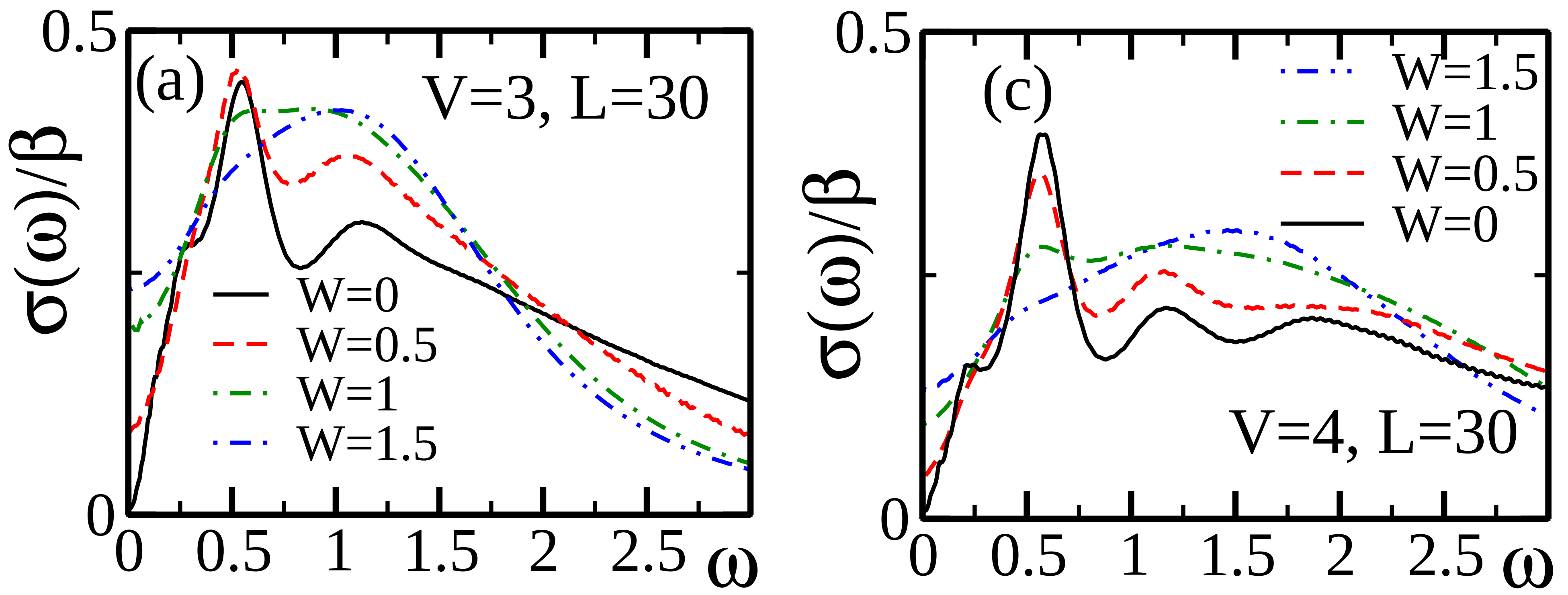
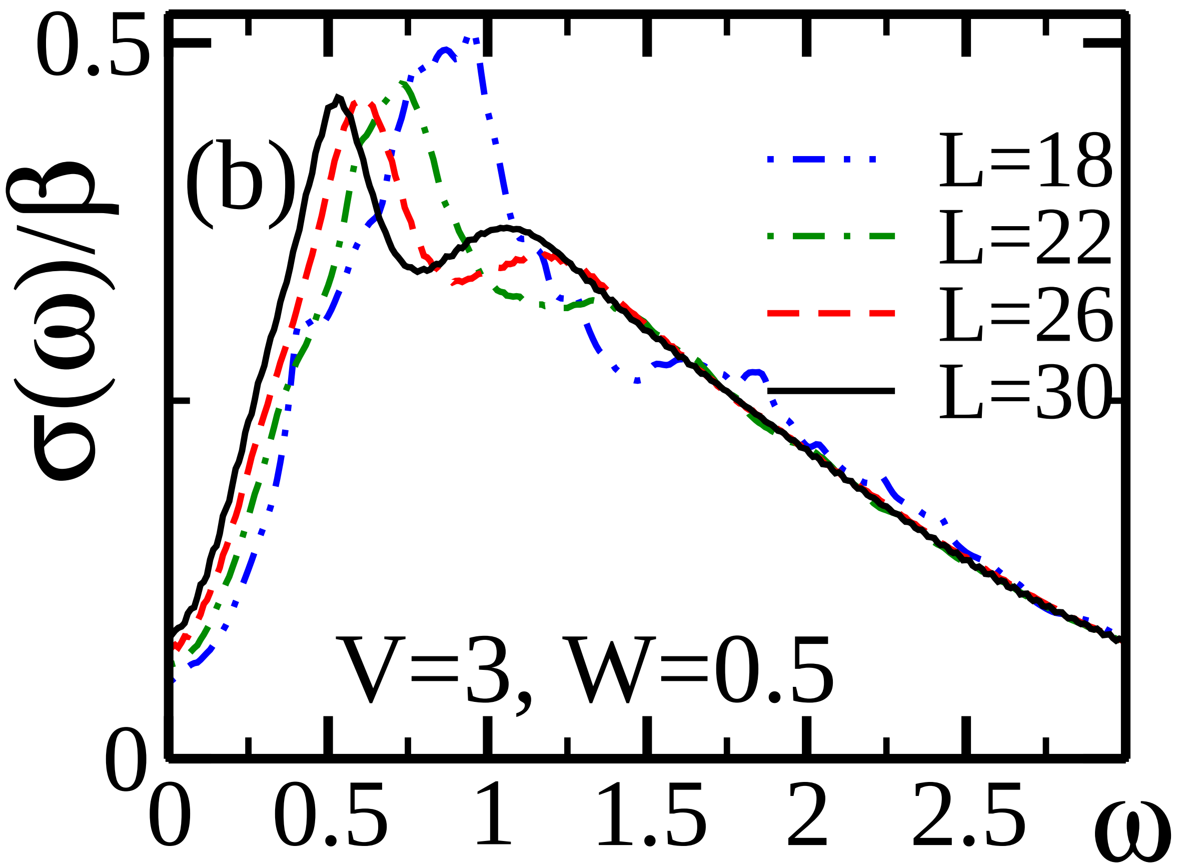


LR for various $V+$ small W

Does $\sigma(0)$ vanish for integrable case?



Finite–size effects



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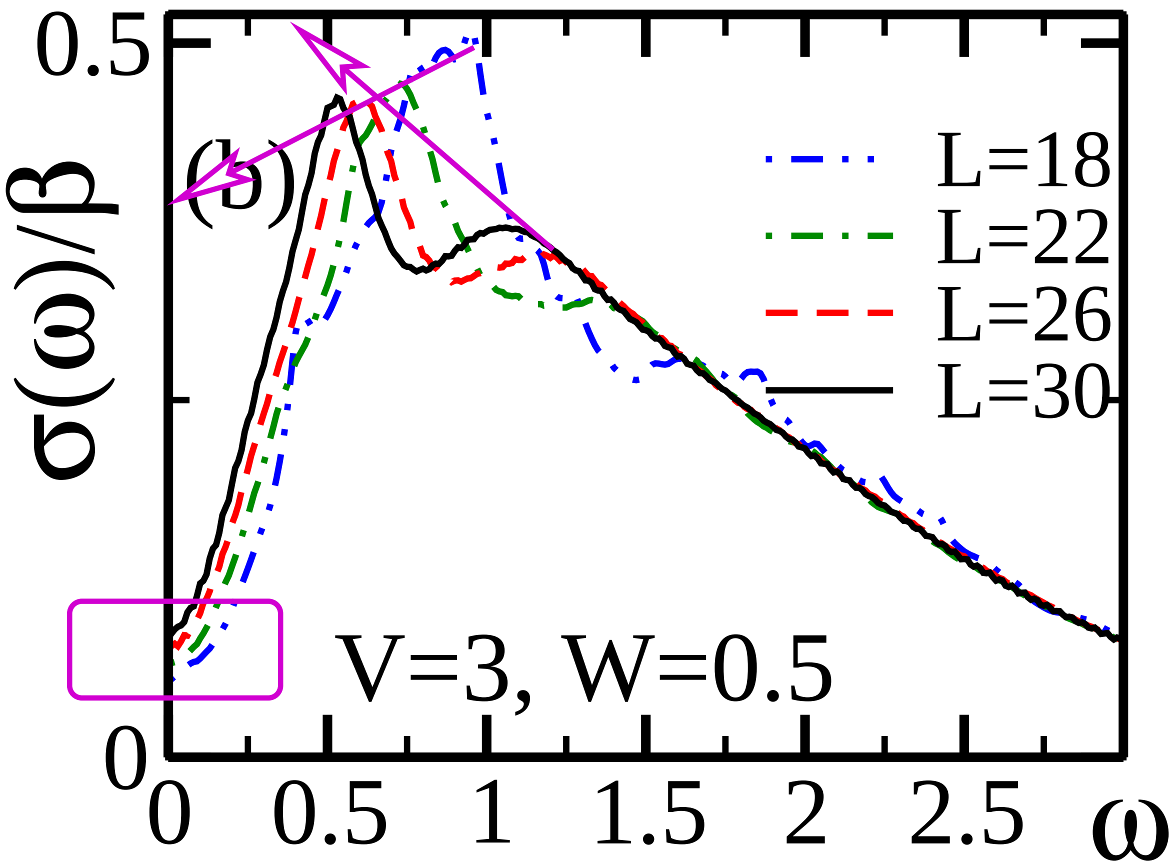
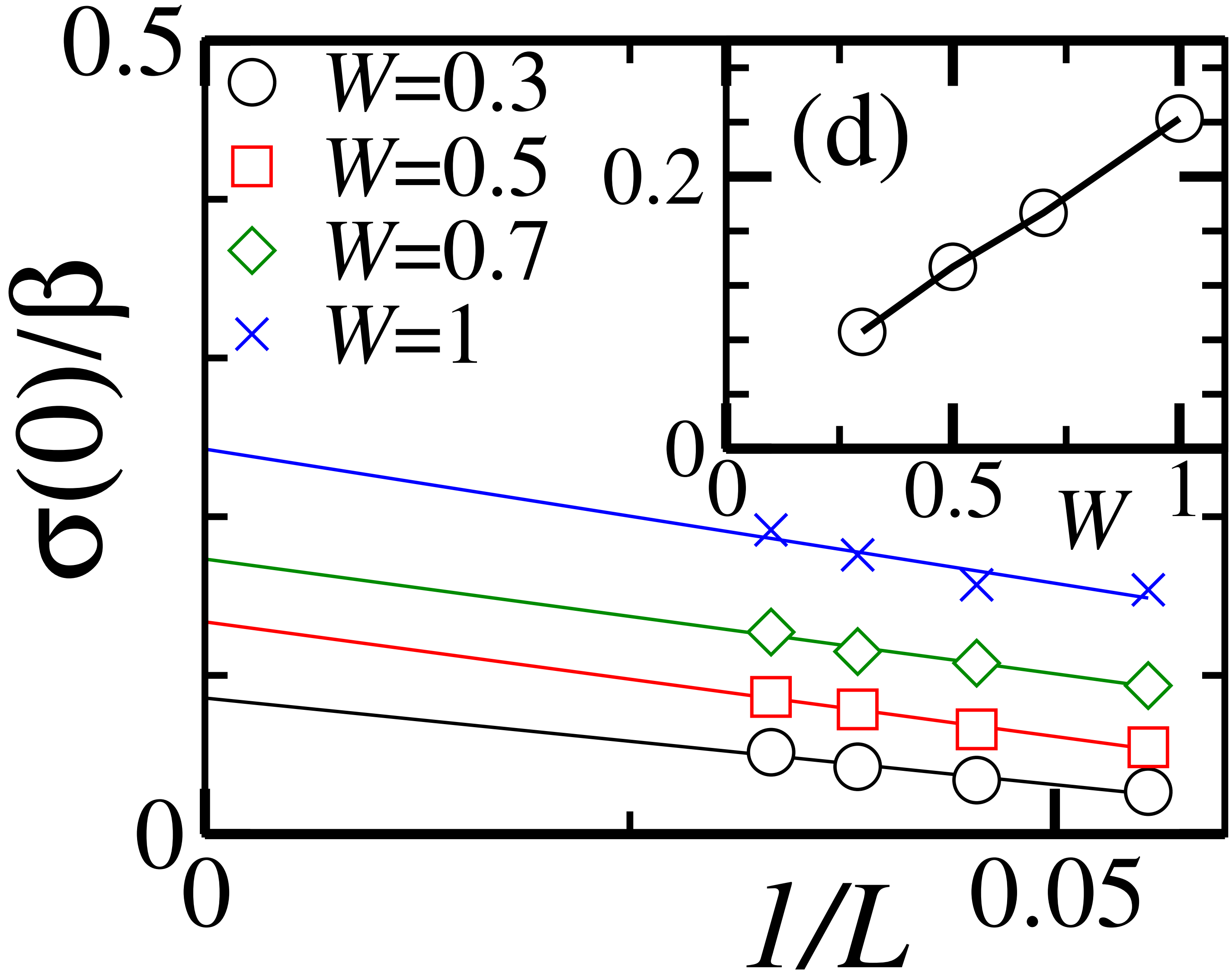


For $\omega > \omega_{FS} \simeq 1.25$ finite–size effects not visible.

Sum rule:: $\int_0^{\omega_{FS}} d\omega\sigma(\omega) + \int_{\omega_{FS}}^{\infty} d\omega\sigma(\omega) = \text{const}$

Finite–size effects



Summary on driving of insulators:

Setup under consideration:

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Interpretation:

- ’Ideal insulator’ with LR regime determined by mechanisms which break integrability

Details in:

M.M., J. Bonča, and P. Prelovšek, Phys. Rev. Lett. **107**, 126601 (2011).

Beyond LR - nonintegrable metal

Expectations: the Joule heating $(\propto F^2) \longrightarrow$ increase of E_k

- LR: $\int d\omega\sigma(\omega) \propto -E_k$
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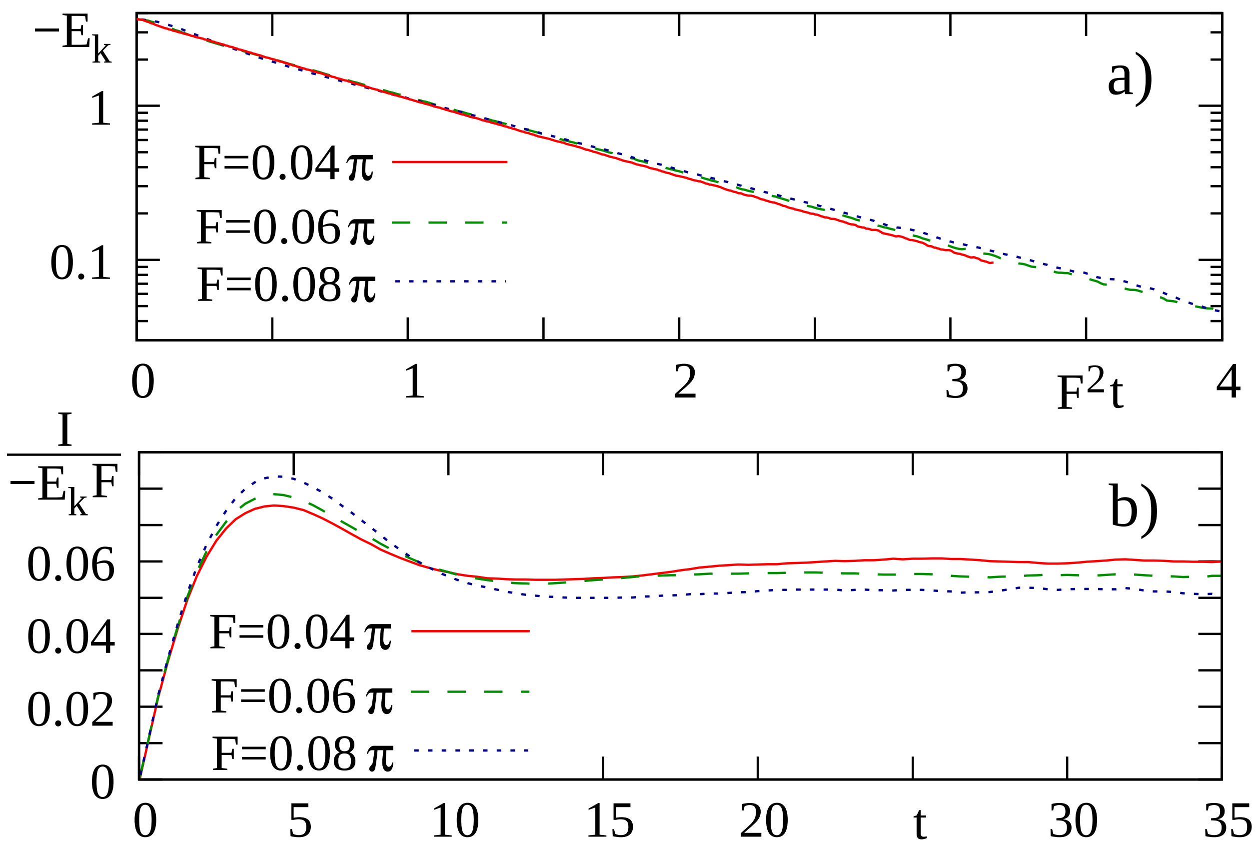


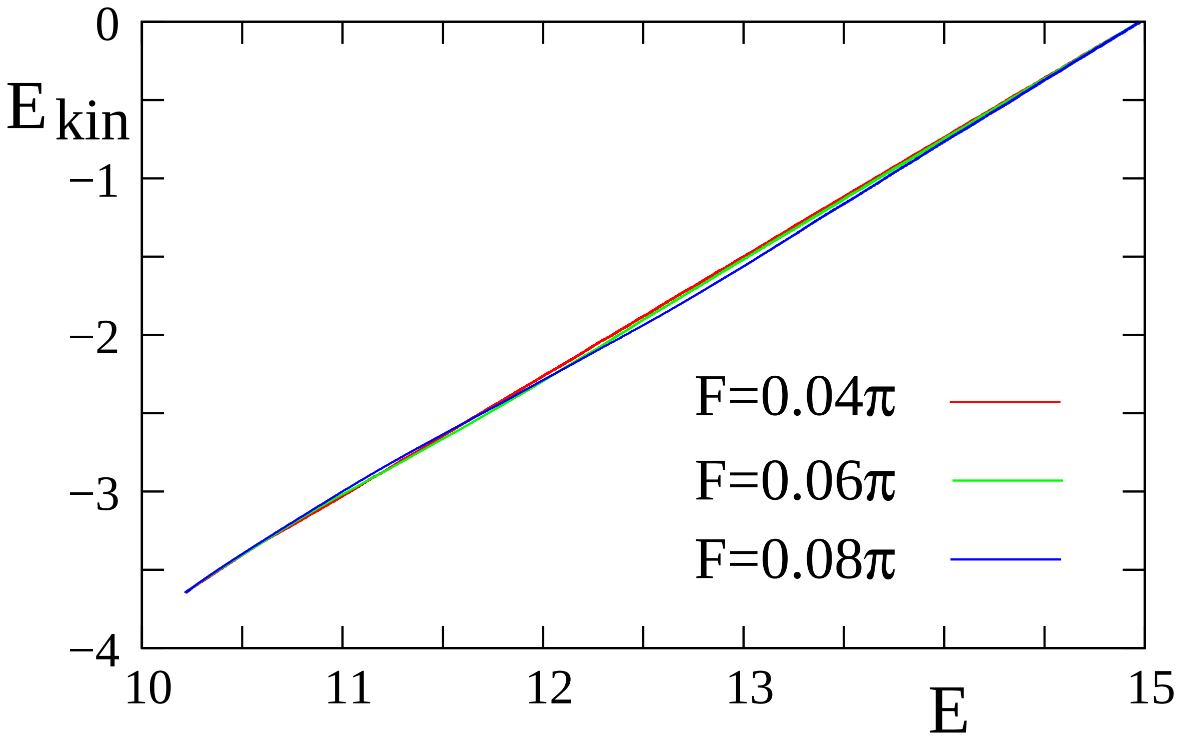
Beyond LR - nonintegrable metal

How to predict strongly nonlinear response without explicit solution of the von Neumann or the time–dependent Schrödinger equations?

Conjecture:

- $I(t) \simeq I_{\rm ER}(t) = \dfrac{E_k(t)}{E_k(0)} I_{\rm LR}(t), \qquad I_{\rm LR}(t) = \int_0^t {\rm d}t' \sigma(t-t') F(t')$

$$\frac{d}{dt} E_k(t) = i\langle [H(t), H_k(t)]\rangle + L\, F(t)\, I(t)$$



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- In the long–time regime:

$I_{\mathrm{ER}}(t) \propto -E_k(t) F$, then $-E_k(t) \propto \exp(-\alpha F^2 t)$ with $\alpha > 0$

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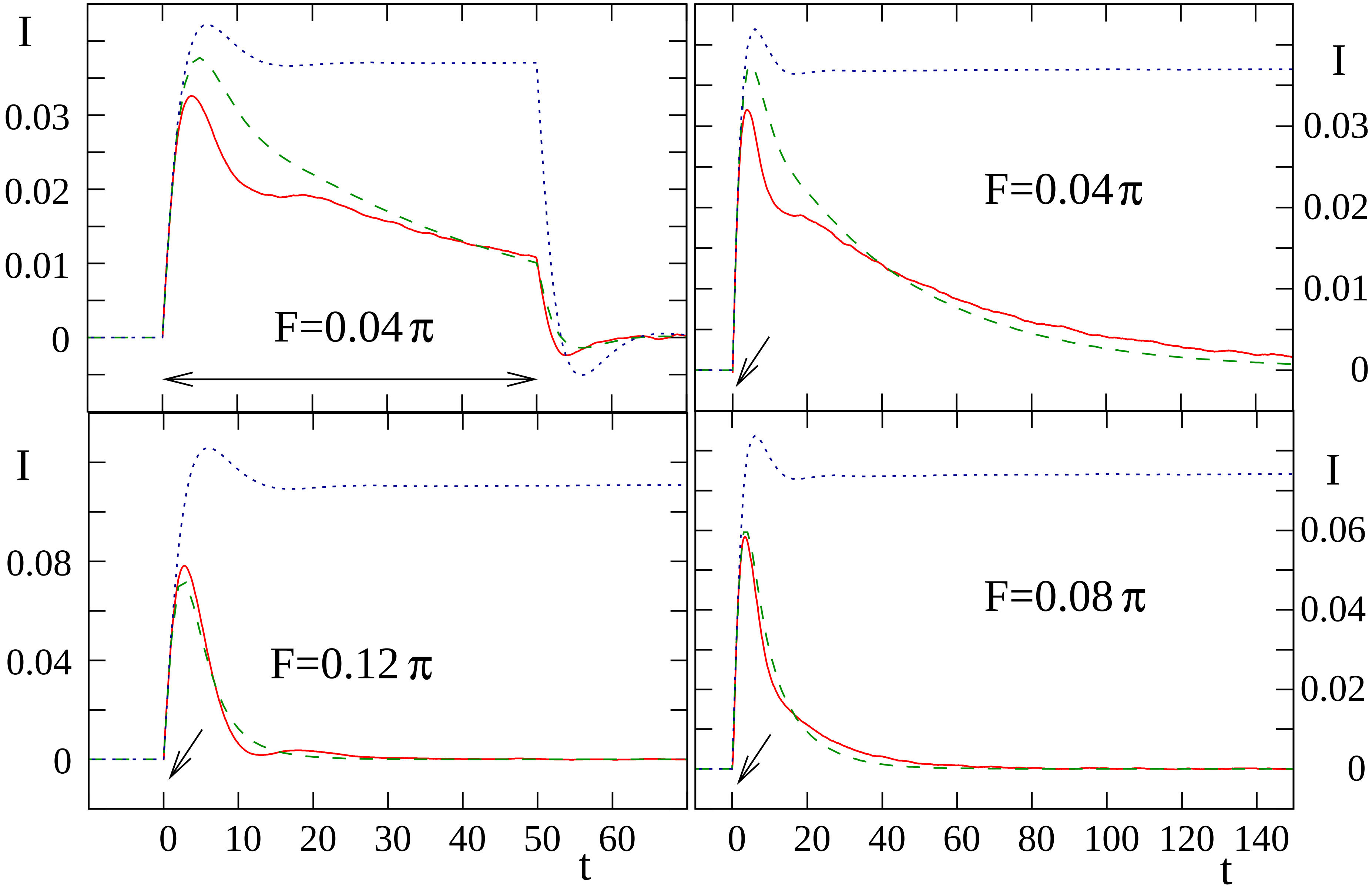
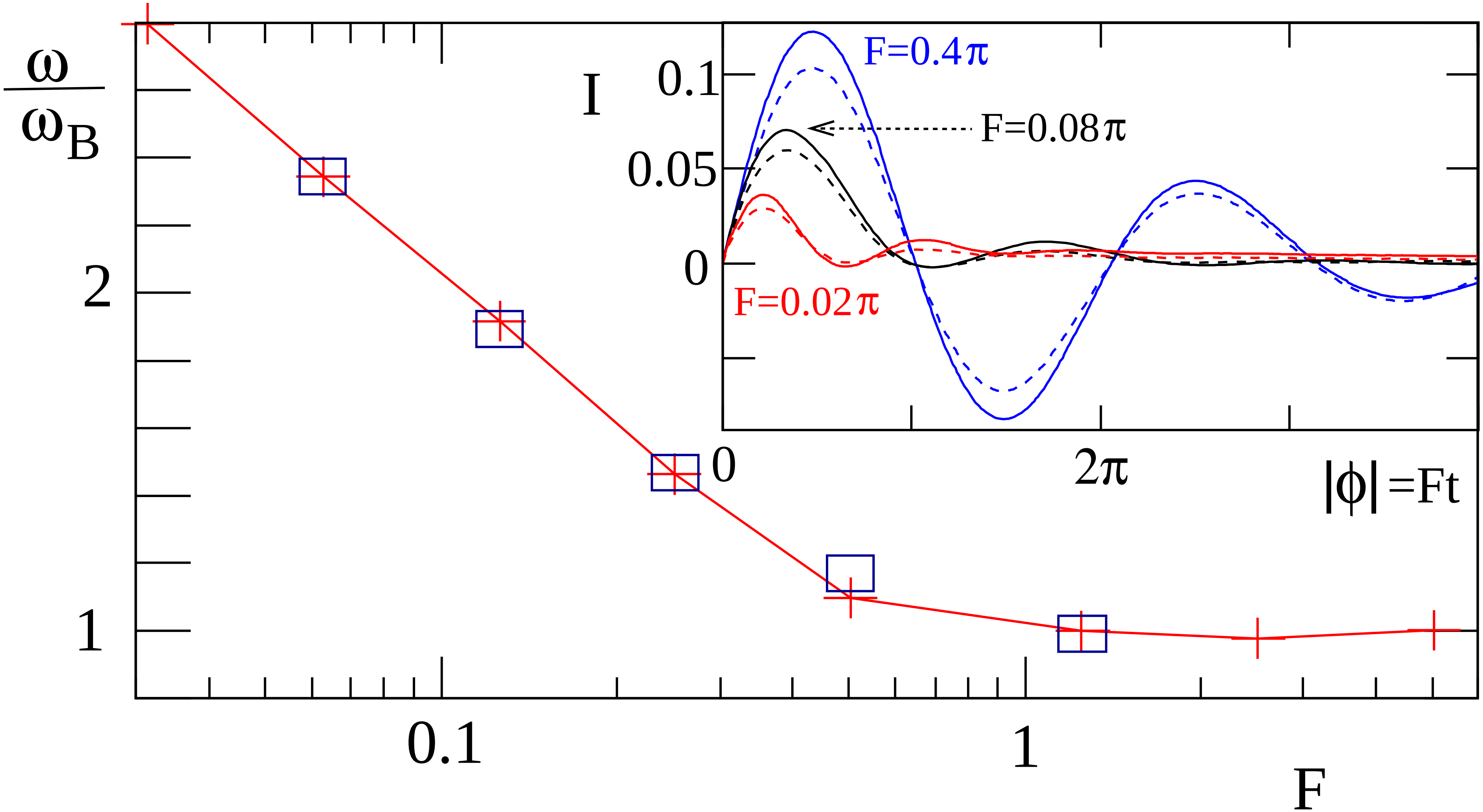
- $$\gamma = \frac{E_k(t) - E_k(0)}{E(t) - E(0)} \simeq \frac{\partial E_k(0)}{\partial T} \left(\frac{\partial E(0)}{\partial T} \right)^{-1}.$$

Numerical check -nonintegrable metal



Integrable metal



Interpretation: charge stiffness at any $T \longrightarrow$ damped Bloch oscillations

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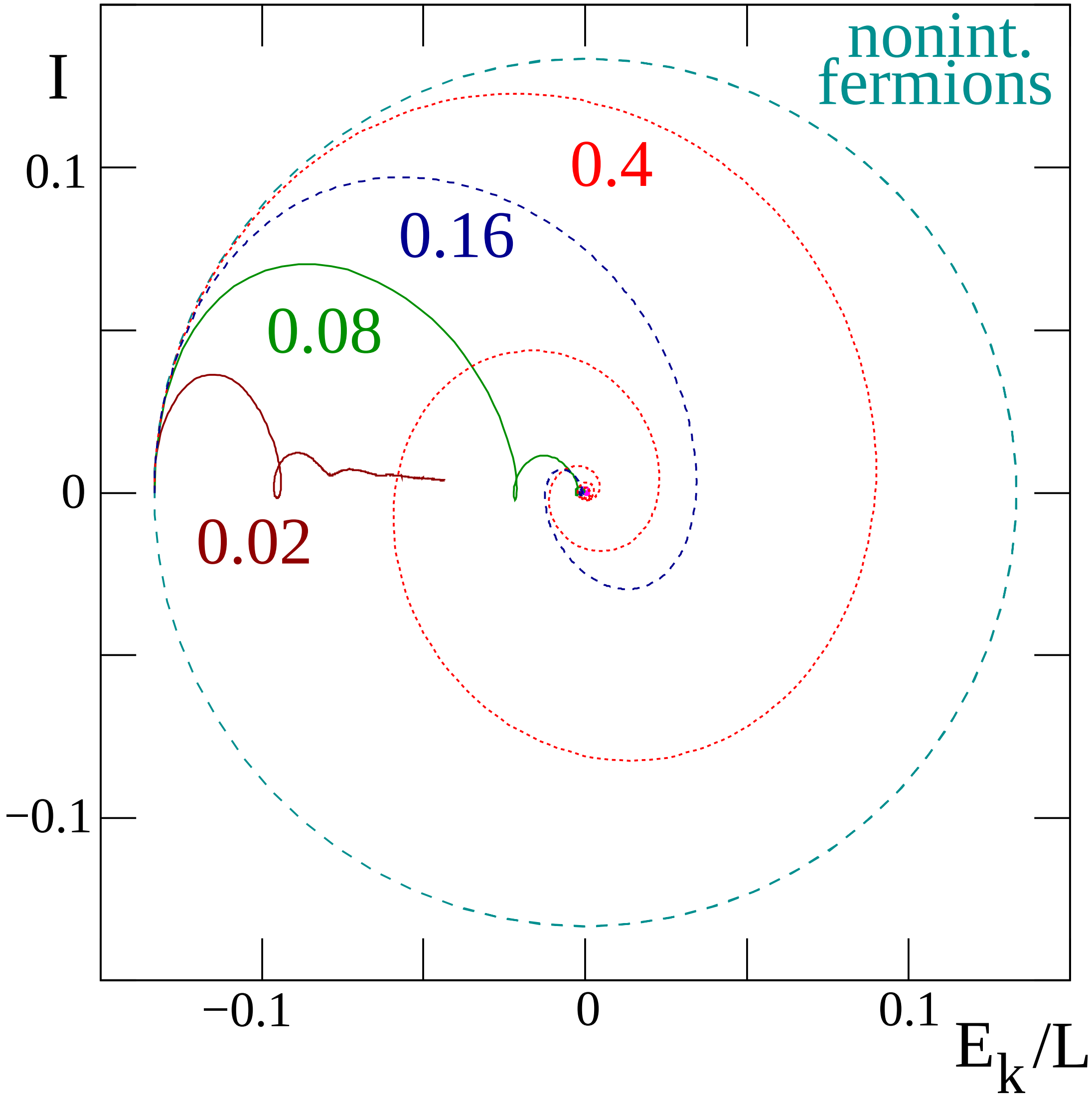

Short time and large field

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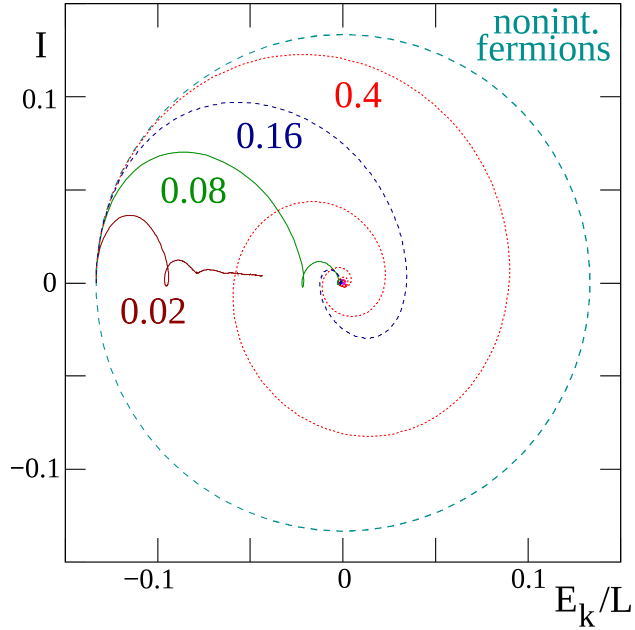

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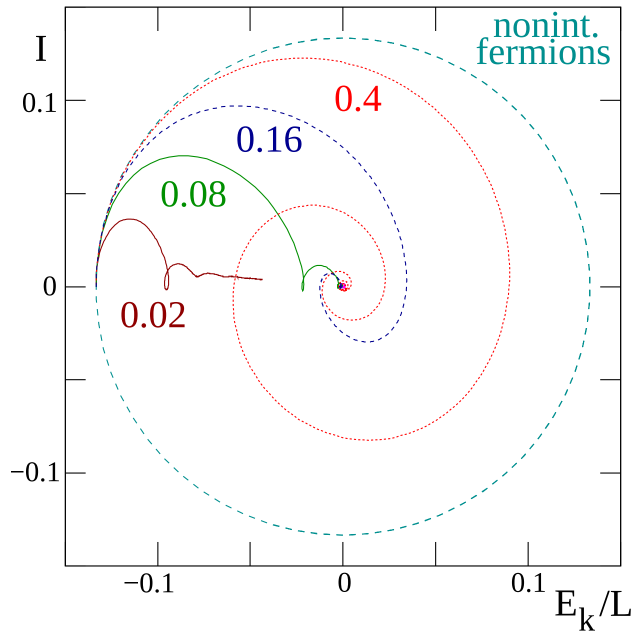

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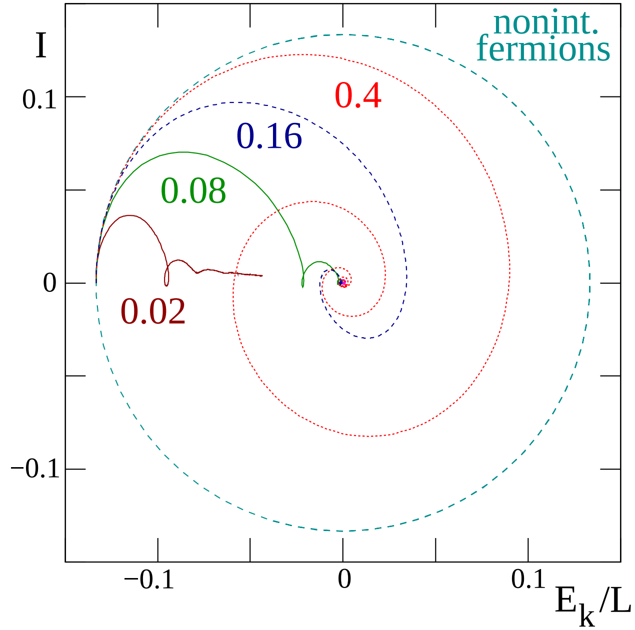

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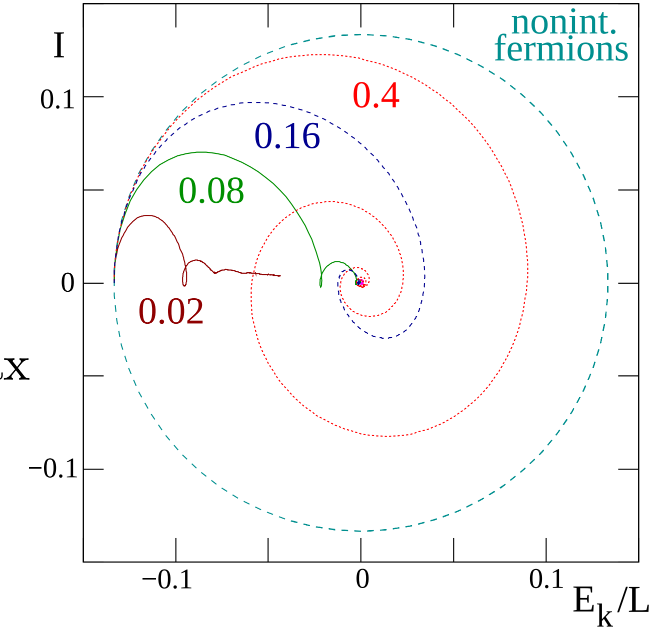

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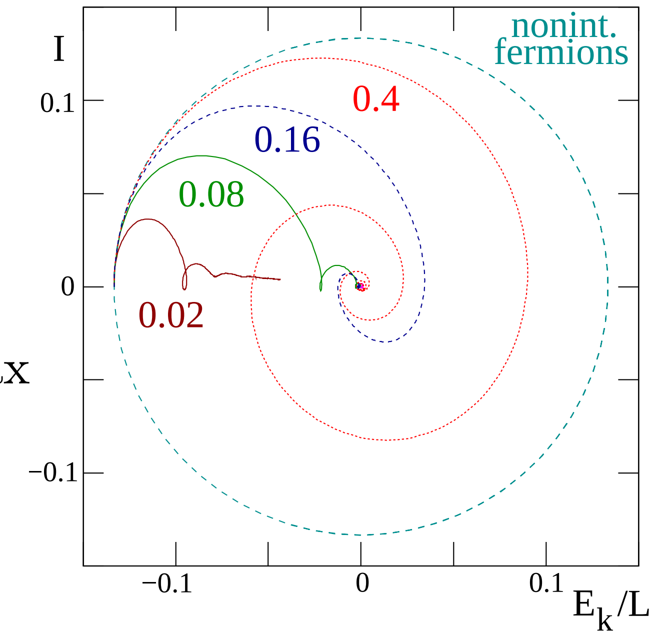

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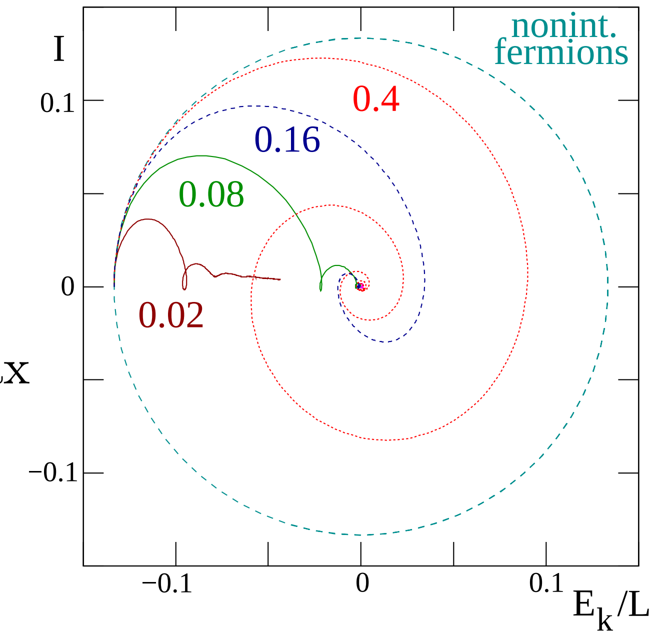

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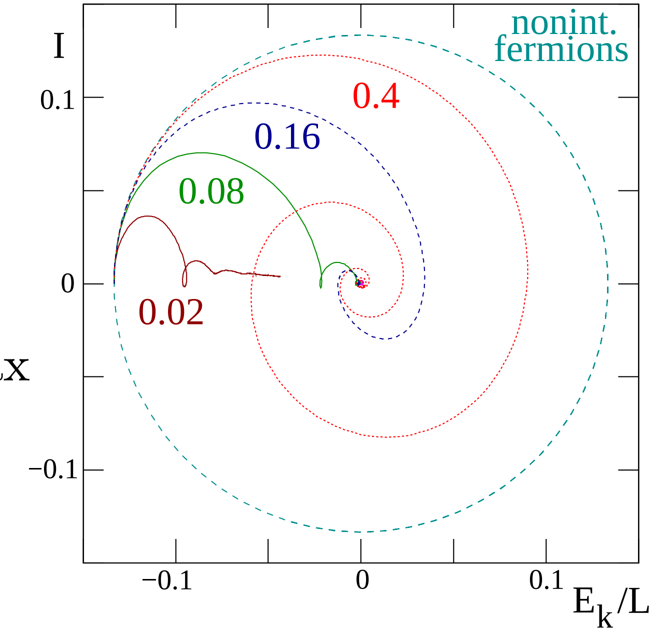

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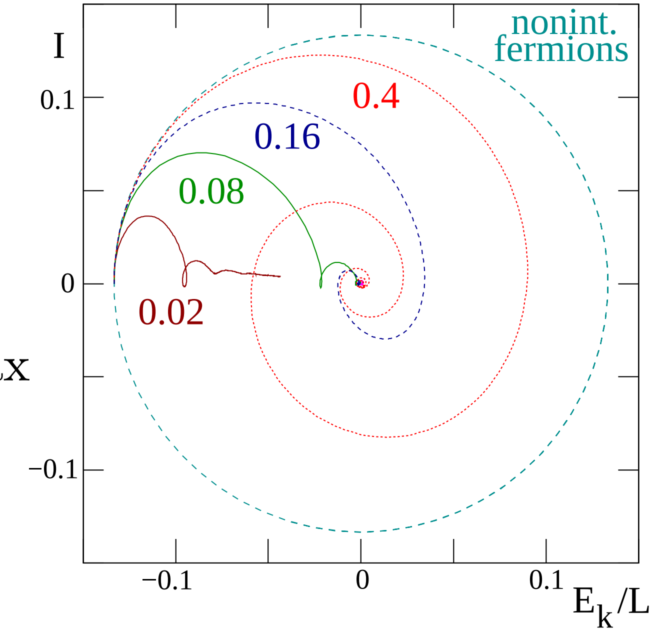

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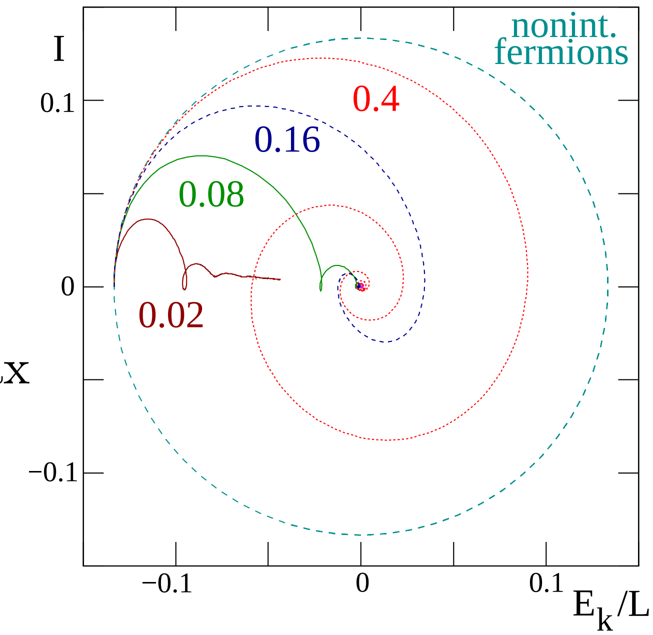
- $\left[\frac{E_k(t)}{L}\right]^2 + I^2(t) \simeq \mathrm{const}$
- for non-interacting particles valid for arbitrary t and $F(t)$



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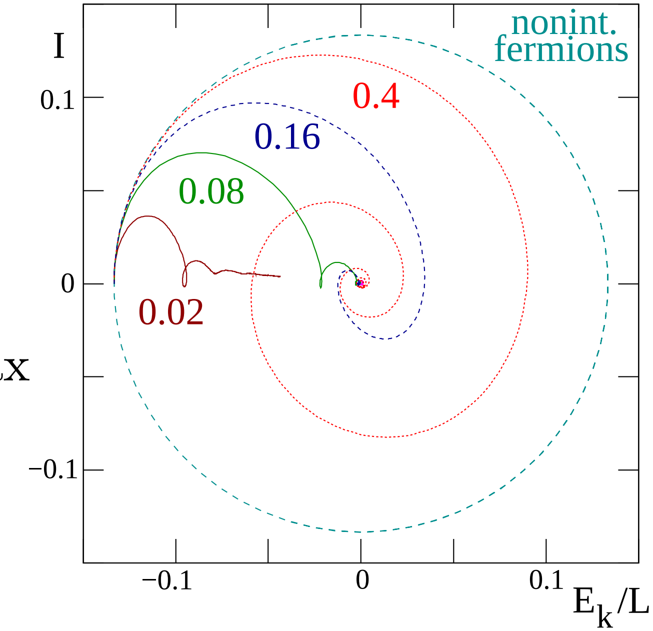
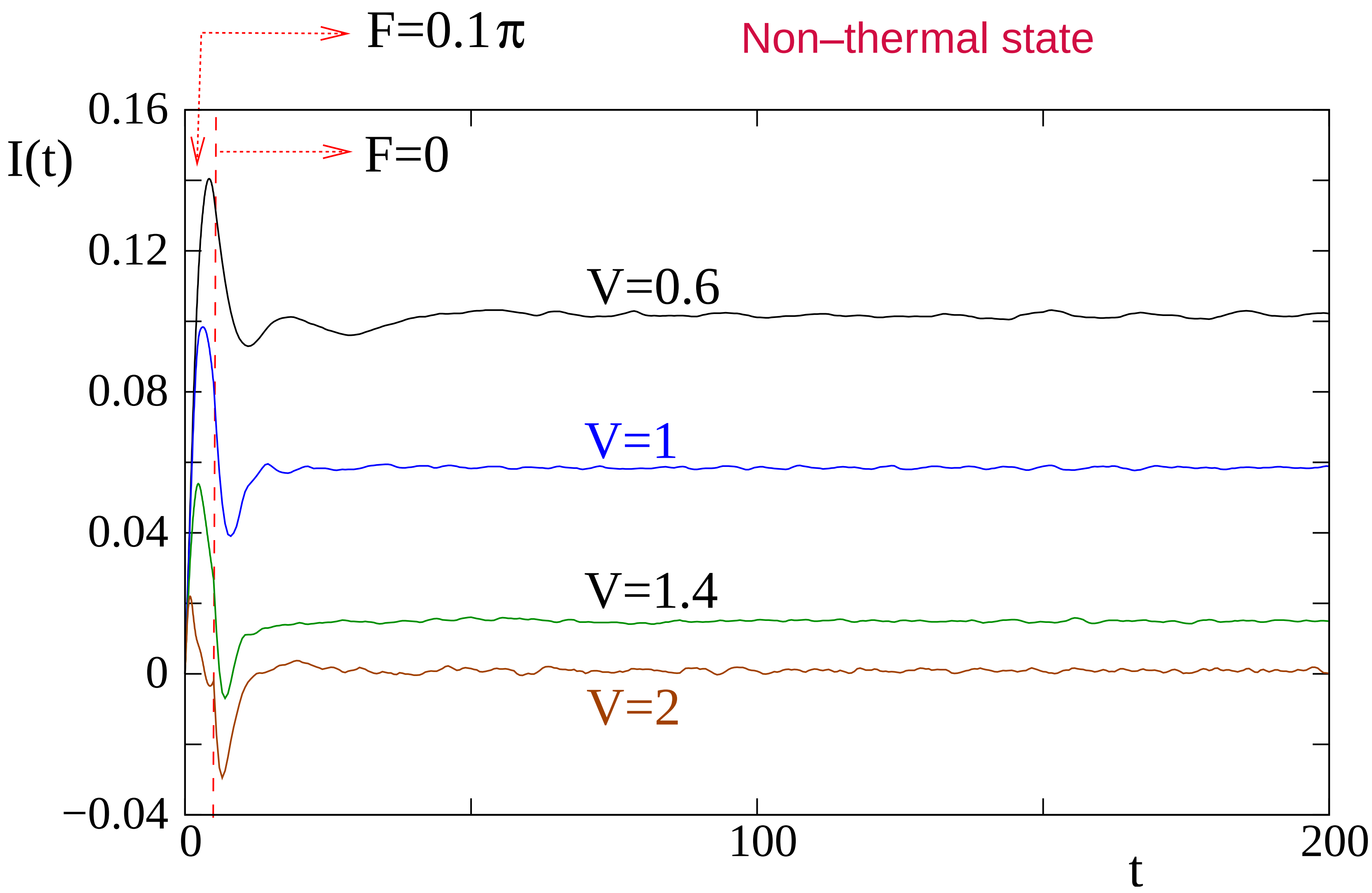
- but eigenvalues of $H(t)$ bounded by $E_{\max}$
- for $t \ll E_{\max}^{-1}$, $\rho(t) \simeq \rho(0)$
- $I(t) \simeq \frac{1}{L}\sum_k 2\sin[k-\phi(t)]\langle c_k^\dagger c_k\rangle_{t=0}$
- $E_k(t) \simeq \sum_k 2\cos[k-\phi(t)]\langle c_k^\dagger c_k\rangle_{t=0}$
- $\left[\frac{E_k(t)}{L}\right]^2 + I^2(t) \simeq \mathrm{const}$
- for non-interacting particles valid for arbitrary t and $F(t)$
- $I(t)$ bounded by initial E_k



Absence of relaxation in integrable case



Summary on driving of metals:

Spinless fermions at half–filling, large (initial) temperature

- different responses of integrable and nonintegrable 1D metals
- nonintegrable: the Joule heating as a dominating nonlinear mechanism for large (initial) T
- nonintegrable: real–time current without formal solution of time–dependent problem
- integrable: the damped Bloch oscillations with (logarithmically) modified frequency

Details in: M.M. and P. Prelovšek, Phys. Rev. Lett. 105, 186405 (2010)

1D t–V model

$$H = H_k + H_I \qquad H_k = -t_h \sum_j \left\{ \mathrm{e}^{i\phi(t)}\, c_{j+1}^\dagger c_j + \mathrm{h}.c. \right\}$$

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$$\begin{aligned}
\frac{d}{dt} E(t) &= L\; F(t)\; I(t) \\
\frac{d}{dt} E_k(t) &= i\langle [H(t), H_k(t)] \rangle + L\; F(t)\; I(t) \\
\frac{d}{dt} I(t) &= i\langle [H(t), J(t)] \rangle - F(t) \frac{E_k(t)}{L}
\end{aligned}$$

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$H_I = 0 \longrightarrow$ the Bloch oscillations (F=const), $I(t) = I_0 \cos(Ft)$.

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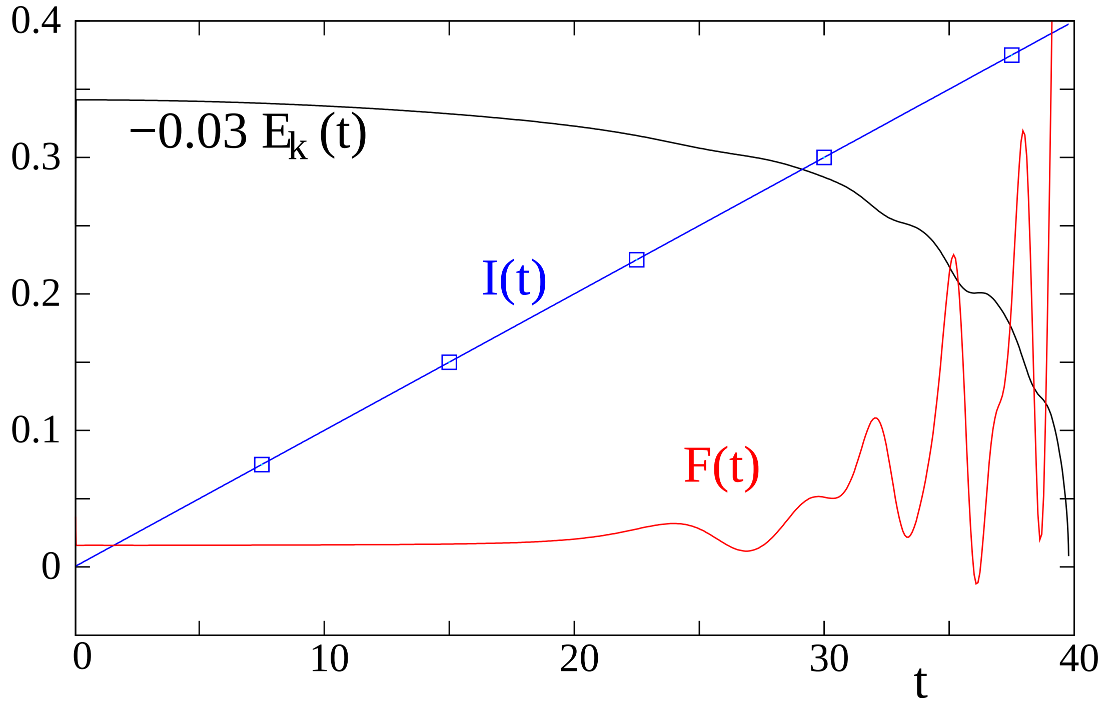
$L = 18, V = 1.4, W = 1$ with the ground state $|\psi(0)\rangle$



Interpretation of finite size–effects

Standard approach for open bc:

$$H = -\sum_j [c^\dagger_{j+1} c_j + \mathrm{h.}c.] + F \sum_j j c^\dagger_j c_j$$

$$c_j = \sum_m J_{j-m}(\frac{2}{F})\ c_m \qquad\qquad H = \sum_m F\ m\ c^\dagger_m c_m$$

$J_m(x)$ - Bessel function of the 1st kind; localization length: $l = \frac{4}{F}$

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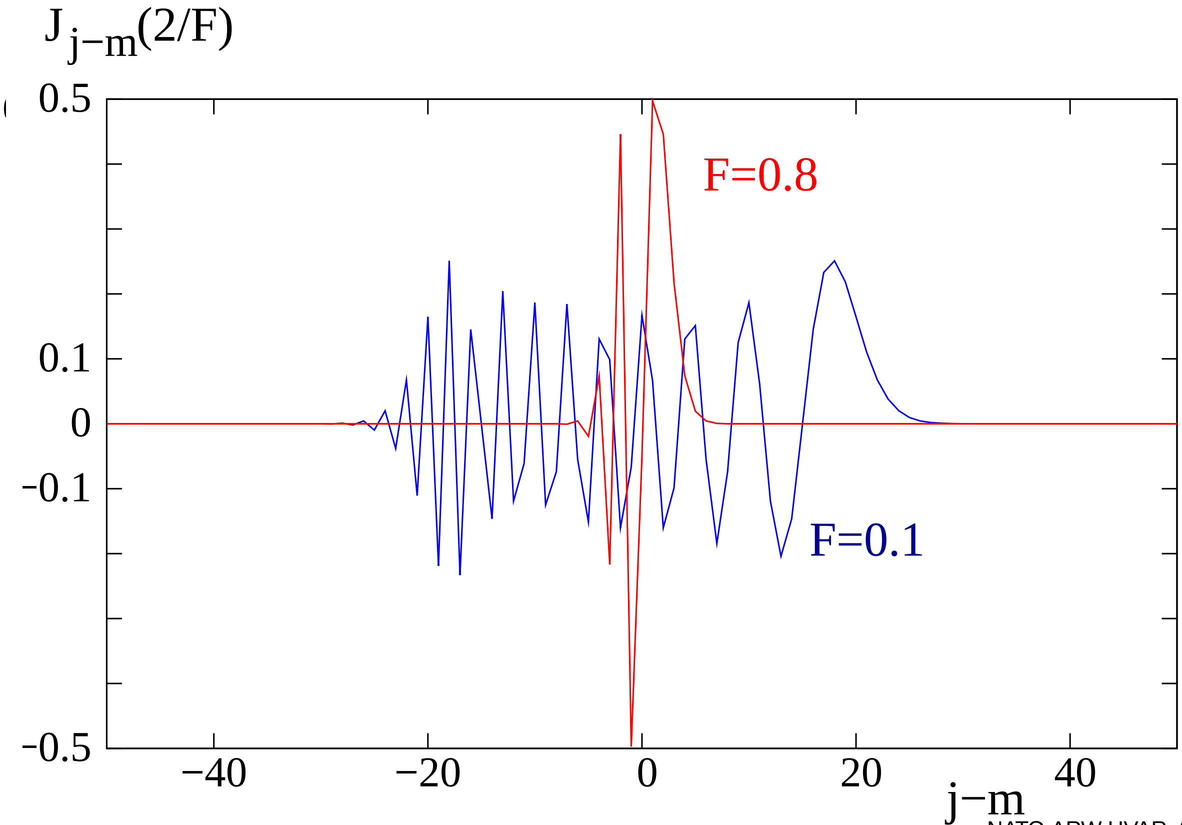
$J_m(x)$ - Bessel function



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Time–dependent flux with $F = \mathrm{const}$, single particle:

$$\dot{E}(t) = LFI(t) \to \Delta E(t) = E(t) - E(0) = FL \int_0^t \mathrm{d}\tau I(\tau)$$

$$I(t) = \frac{1}{L} v(t) \to r(t) = L \int_0^t \mathrm{d}\tau I(\tau) = \Delta E(t)/F < \frac{4}{F}$$